

REFLECTION PRINCIPLES FOR CLASS GROUPS

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ABSTRACT. We present several new examples of reflection principles which apply to both class groups of number fields and picard groups of curves over $\mathbb{P}^1/\mathbb{F}_p$. This proves a conjecture of Lemmermeyer [3] about equality of 2-rank in subfields of A_4 , up to a constant not depending on the discriminant in the number field case, and exactly in the function field case. More generally we prove similar relations for subfields of a Galois extension with group G for the cases when G is S_3 , S_4 , A_4 , D_{2l} and $\mathbb{Z}/l\mathbb{Z} \rtimes \mathbb{Z}/r\mathbb{Z}$. The method of proof uses sheaf cohomology on 1-dimensional schemes, which reduces to Galois module computations.

INTRODUCTION

In this paper we look at the problem of relating the size of the l -torsion in the class groups of two distinct number fields. We let $\mathrm{rk}_l Cl(K) = \dim_{\mathbb{F}_l} Cl(K)[l]$.

Describing the size of the l -torsion of the class group of a number field is in general a hard problem. There are special cases where one can say something about $\mathrm{rk}_l Cl(K)$. For example it is easy to show for any quadratic field K that $\mathrm{rk}_2 Cl(K) = r - 1$, where r is the number of primes ramified in K . One source of theorems describing l -torsion in class groups is Iwasawa theory, which gives formulas for $\mathrm{rk}_l Cl(K)$ for K lying in some tower of number fields, in terms of certain invariants depending on the base field. A very strong asymptotic conjecture on class group order is the following due to Zhang [8]:

Conjecture. *For any number field K let $n = [K : \mathbb{Q}]$ and let $\epsilon > 0$. Then*

$$|Cl(K)[l]| \ll_{\epsilon, n, l} D_K^\epsilon.$$

That is the l -torsion in number fields of a fixed degree grows slower than any power of their discriminant. Some work has been done in this direction by Ellenberg and Venkatesh [2] by combining reflection principles with analytic techniques.

A slightly different kind of problem is that of relating l -rank in class groups of two different number fields. Such statements are often called reflection principles. There are many such statements known and we will give a very brief overview of some of these. For a more exhaustive exposition we refer the reader to [3].

The following is referred to as the Scholz reflection theorem [7]:

Proposition 1. *Let $D > 1$ be square-free. Let $K = \mathbb{Q}(\sqrt{-3D})$ and $F = \mathbb{Q}(\sqrt{D})$. Then $\mathrm{rk}_l Cl(F) \leq \mathrm{rk}_l Cl(K) \leq \mathrm{rk}_l Cl(F) + 1$.*

And a generalization of the above due to Leopoldt [7]:

Proposition 2. *Let $K = \mathbb{Q}(\zeta_l)$. Let $Cl(K)^\pm$ denote the positive and negative eigenspace of $Cl(K)$ under the action of complex conjugation in $\text{Gal}(K/\mathbb{Q})$. Then $\text{rk}_l Cl(K)^+ \leq \text{rk}_l Cl(K)^-$.*

Another result of this form is due to Bolling for subfields of dihedral extensions [1]:

Proposition 3. *Let L be a number field with $\text{Gal}(L/\mathbb{Q}) = D_{2l}$, where p is an odd prime, and let K be any of its subfields of degree p . Assume that the quadratic subfield F of L is complex. Then*

$$\text{rk}_l Cl(F) - 1 \leq \text{rk}_l Cl(K) \leq \frac{(l-1)}{2} (\text{rk}_l Cl(F) - 1).$$

Recently Tsimerman proved a result about class groups of algebraic tori [6], which he applied to derive reflection principles similar to the above. It has the downside that the error term depends on the discriminant of the field:

Proposition 4. *Let L be a number field with $\text{Gal}(L/\mathbb{Q}) = S_4$ or A_4 . Let K_1 be a quartic subfield, and K_2 be its cubic resolvent. Then $\text{rk}_2 Cl(K_1) = \text{rk}_2 Cl(K_2) + O_\epsilon(D_L^\epsilon)$.*

Lemmermeyer conjectures [3] that in this case the following (sharp) bound holds:

Conjecture 5. *In the above $\text{rk}_2 Cl(K_1) - 2 \leq \text{rk}_2 Cl(K_2) \leq \text{rk}_2 Cl(K_1)$.*

We will build on Tsimerman's approach and restate the problem using sheaf cohomology. This lets us prove reflection principles in the general setting of picard groups of 1-dimensional schemes, which can then be applied to derive reflection principles for both class groups of number fields and picard groups of function fields. Among the results are an improvement of Proposition 4 by removing the dependence on the discriminant in favor of a bound similar to that in the other statements above, which proves Conjecture 5 up to an error of $O(1)$.

We summarize our results below, and state them in more detail in the main part of the paper.

Theorem 6. *Let S be either $\text{Spec} \mathbb{Z}[1/l]$ or $\mathbb{P}^1/\mathbb{F}_p$ where p and l are distinct primes and let $\eta = \text{spec} F$ be the generic point of S . Consider two finite covers $\pi_i : X_i \rightarrow S$ for $i = 1, 2$ with generic points $\text{Spec} K_i$. Let L/F be a Galois extension containing the K_i and $G = \text{Gal}(L/F)$. We will always make the assumption that L/K_2 is unramified. Suppose we have one of the following cases:*

- (1) *Let $l = 3$. Let $G = S_3$. Let K_1 be a cubic subfield and K_2 its quadratic resolvent.*
- (2) *Let $l = 2$. Let $G = A_4$ or S_4 . Let K_1 be a quartic subfield and K_2 its cubic resolvent.*

Then we have the bound

$$\text{rk}_l Cl(K_2) - C_1 \leq \text{rk}_l Cl(K_1) \leq \text{rk}_l Cl(K_2) + C_2$$

where the C_i are constants depending on the case and on whether $\mu_l \subset \mathbb{F}_p$ if $S = \mathbb{P}^1$, and can be computed explicitly.

Computing the above constants explicitly yields the following Corollary which proves Conjecture 5 up to an error of $O(1)$.

Corollary 7. *Let L/\mathbb{Q} be a Galois extension with $\text{Gal}(L/\mathbb{Q}) = A_4$. Let K_1 be a quartic subfield and K_2 its cubic resolvent. Then if L is real we have*

$$\text{rk}_2 \text{Cl}(K_2) - 10 \leq \text{rk}_2 \text{Cl}(K_1) \leq \text{rk}_2 \text{Cl}(K_2) + 10$$

and if L is complex we have

$$\text{rk}_2 \text{Cl}(K_2) - 8 \leq \text{rk}_2 \text{Cl}(K_1) \leq \text{rk}_2 \text{Cl}(K_2) + 12.$$

Additionally in the function field case we obtain Conjecture 5 exactly:

Corollary 8. *Let $L/\mathbb{F}_p(T)$ be a Galois extension with $\text{Gal}(L/\mathbb{F}_p(T)) = A_4$. Let K_1 be a quartic subfield and K_2 its cubic resolvent. Then*

$$\text{rk}_2 \text{Pic} C_2 - 2 \leq \text{rk}_2 \text{Pic} C_1 \leq \text{rk}_2 \text{Pic} C_2.$$

We also obtain reflection principles in a new case, when $\text{Gal}(L/\mathbb{Q}) = \mathbb{Z}/l\mathbb{Z} \rtimes \mathbb{Z}/r\mathbb{Z}$, which can be thought of as a generalization of the dihedral case of Proposition 3. Let the notation be the same as in Theorem 6.

Theorem 9. *With the same assumptions as in Theorem 6, suppose we have one of the following cases:*

- (1) *Let l be an odd prime. Let $G = D_{2l}$. Let K_1 be any of the subfields of degree l and K_2 be the quadratic subfield.*
- (2) *Let l, r be odd primes with $r \equiv 1 \pmod{l}$. Let $G = \mathbb{Z}/l\mathbb{Z} \rtimes \mathbb{Z}/r\mathbb{Z}$. Let K_1 be any of its subfields of degree l and K_2 be the subfield of degree r .*

Then we have the bound

$$C_1 \text{rk}_l \text{Cl}(K_2) + C_2 \leq \text{rk}_l \text{Cl}(K_1) \leq C_3 \text{rk}_l \text{Cl}(K_2) + C_4$$

where the C_i are constants depending on the case and on whether $\mu_l \subset \mathbb{F}_p$ if $S = \mathbb{P}^1$, and can be computed explicitly.

We emphasize that the constants do not depend on the discriminant of the field. The constants differ in each case and between the number field and function field setting. We will consider each case separately and compute them explicitly.

We also note that even though S_3 specializes D_{2l} which specializes $\mathbb{Z}/l\mathbb{Z} \rtimes \mathbb{Z}/r\mathbb{Z}$, we state the cases separately since our constants improve with each specialization.

Comments and further directions: It is likely that the bounds we obtain can be improved by a more careful consideration of the morphisms in our long exact sequences. For example in Lemma 22 determining s and t would require computing the maps $H^1(M') \rightarrow H^2(\mu_3)$ and $H^1(\mu_3) \rightarrow H^1(N')$ respectively. One possible way of doing this is explicitly computing the maps in terms of Čech cohomology. However this approach can become computationally tedious.

We note that the bounds we obtain are generally sharper in the function field case, and especially when $\mu_l \not\subset \mathbb{F}_p$. In the number field case precision is lost as a result of inverting the prime l , since this increases the size of the unit group and forces us to work with the class group away from the primes above l , both of which play a key role in our computations.

It would be interesting to find more examples of field extensions for which such reflection principles hold. Our method applies more generally to any subfields of a Galois extension provided there exists a series of exact sequences relating the two modules generated by the embeddings of each subfield, along with an additional condition. The procedure we have used for computing such sequences is to find the Jordan-Holder decomposition of each module. By relating the modules we mean roughly that the composition factors in their Jordan-Holder decompositions are the same. The additional condition is that the sequences remain exact upon taking invariants by certain subgroups of the Galois group (we refer to the remainder of the paper for details regarding this). It would be interesting to gain a better understanding of when this type of situation occurs.

1. PRELIMINARY RESULTS

We start by defining the notation and developing some basic results which will be used throughout each of the examples. By H^i we will always mean H_{et}^i .

1.1. Schemes and Picard groups. Let l be a prime. Let S be the scheme equal to either $\text{Spec}\mathbb{Z}[1/l]$ or $\mathbb{P}_{\mathbb{F}_p}^1$ where p is a prime distinct from l . Let $\eta = \text{Spec}F$ be the generic point of S and denote by $g : \eta \rightarrow S$ the inclusion. Consider a scheme $\pi : X \rightarrow S$ which is a finite degree n cover of S . When $S = \mathbb{P}_{\mathbb{F}_p}^1$ we will let X be a complete connected smooth curve and when $S = \text{Spec}\mathbb{Z}[1/l]$ we let X be of the form $\text{Spec}\mathcal{O}_K[1/l]$ for some number field K . Let $Z \subset S$ be the finite set of points above which π is ramified. Let $\text{Spec}K$ be the generic point of X , so $[K : F] = n$.

Consider the Kummer sequence of etale sheaves on X

$$1 \longrightarrow \mu_l \longrightarrow \mathbb{G}_m \longrightarrow \mathbb{G}_m \longrightarrow 1$$

We assume that X has residue characteristic coprime to l at each point, so that the Kummer sequence is exact. Taking cohomology of this sequence and noting that $H^1(X, \mathbb{G}_m) = \text{Pic}(X)$ gives

$$\begin{array}{ccccccc} 1 & \longrightarrow & \mathcal{O}_X(X)^\times[l] & \longrightarrow & \mathcal{O}_X(X)^\times & \longrightarrow & \mathcal{O}_X(X)^\times \\ & & & & \searrow & & \\ & & H^1(X, \mu_l) & \xleftarrow{\quad} & \text{Pic}(X) & \longrightarrow & \text{Pic}(X) \end{array}$$

Let $\mathcal{L} = \pi_*\mu_l$. This is a finite locally constant sheaf on $S \setminus Z$. Since push-forward of sheaves by finite morphisms is exact and preserves injectives, we have $H^1(X, \mu_l) = H^1(S, \mathcal{L})$. We summarize this as

Lemma 10. *Let $\pi : X \rightarrow S$ be a finite cover and let $\mathcal{L} = \pi_*\mu_l$. Then*

$$\text{Pic}(X)[l] \cong H^1(S, \mathcal{L}) / (\mathcal{O}_X(X)^\times / l\mathcal{O}_X(X)^\times).$$

In all cases we will consider $\mathcal{O}_X(X)^\times / l\mathcal{O}_X(X)^\times$ can be computed explicitly and so $H^1(S, \mathcal{L})$ will be the main object of interest.

1.2. Sheaves and Galois modules. Next we recall some basics about finite locally constant sheaves. We continue with the same notation as above. Let $U = S \setminus Z$. Note Z is closed, so U is open, and let $j : U \rightarrow S$ be the open immersion. Fix $\bar{\eta} = \text{Spec} \bar{F}$ a geometric point above the generic point $\eta = \text{Spec} F$ of S . As above let $\mathcal{L} = \pi_* \mu_l$.

Recall that there is a category equivalence between finite $\pi_1(U, \bar{\eta})$ -modules, finite étale schemes over U , and finite locally constant sheaves of abelian groups on U . It says that a finite locally constant sheaf \mathcal{F} on U is represented by some scheme Y finite étale over U whose geometric points above $\bar{\eta}$ are a $\pi_1(U, \bar{\eta})$ -module.

The following facts are standard and can be found in [4].

Lemma 11. *Let M be a G_F -module representing a sheaf \mathcal{F} on η . Suppose the action of G_F factors through $\text{Gal}(L/F)$ for some finite extension L . Let $V \subset S$ be the set of points which are unramified in L . Then \mathcal{F} extends to a finite locally constant sheaf on V represented by M with an action of $\pi_1(V, \bar{\eta})$.*

Lemma 12. *Let \mathcal{F} be a finite locally constant sheaf on U represented by the $\pi_1(U, \bar{\eta})$ -module M . Then for any point $z \in S \setminus U$ we have $(j_* \mathcal{F})_{\bar{z}} \cong M^{I_z}$ as a D_z/I_z -module, where I_z is the inertia group at z .*

Remark 13. By the above Lemmas since $\mathcal{L}|_U$ is finite locally constant it will be represented on U by its stalk $\mathcal{L}_{\bar{\eta}}$, and since \mathcal{L} is itself the pushforward of the finite locally constant sheaf μ_l , it can be shown that $j_*(\mathcal{L}|_U) = \mathcal{L}$. Thus to describe the stalk $\mathcal{L}_{\bar{z}}$ for any $z \in Z$ we take in I_z invariants of $\mathcal{L}_{\bar{\eta}}$.

Next we use our definition of \mathcal{L} to give a more explicit description of M . Recall that $[K : F] = n$.

Lemma 14. *Let M be the G_F -module representing the stalk $\mathcal{L}_{\bar{\eta}}$. Let $\sigma_1, \dots, \sigma_n$ be the embeddings of K into \bar{F} . Let L be a finite extension of F containing the normal closure of K in \bar{F} as well as $\mu_l(\bar{F})$. Then $M \cong \mathbb{Z}/l\mathbb{Z} \langle \sigma_1, \dots, \sigma_n \rangle$ and the action of G_F factors through $\text{Gal}(L/F)$.*

Proof. We claim $\mathcal{L}_{\bar{\eta}} \cong \mathcal{L}|_{\eta}(\text{Spec} L)$ as G_F -modules. By definition $\mathcal{L}(\text{Spec} L) = \mu_l(L \otimes_F K)$. There is an isomorphism $L \otimes_F K \cong \prod_{i=1}^n L$ since L contains the Galois closure of K . So $\mathcal{L}(\text{Spec} L) = \mu_l(\text{Spec}(\prod_{i=1}^n L)) \cong \prod_{i=1}^n \mathbb{Z}/l\mathbb{Z}$, and furthermore G_F acts on $L \otimes_F K$ by acting on each coordinate which under the above isomorphism translates into an action on each coordinate which also permutes the coordinates in correspondence with the embeddings of K in \bar{F} . Clearly this action factors through $\text{Gal}(L/F)$. It is also clear from this that $\text{Spec} L$ trivializes \mathcal{L} . \square

We now turn our attention towards the main goal of the paper. Consider two schemes $\pi_i : X_i \rightarrow S$ for $i = 1, 2$ of the form described at the beginning of this section, with generic point $\text{Spec} K_i$. Let $\mathcal{L}_i = (\pi_i)_* \mu_l$. We want to find a relationship between $\text{rk}_l \text{Pic}(X_1)$ and $\text{rk}_l \text{Pic}(X_2)$, and we will do this by relating $\text{rk}_l H^1(S, \mathcal{L}_1)$ and

$\mathrm{rk}_l H^1(S, \mathcal{L}_2)$ and using Lemma 10. The latter will be done by constructing a family of exact sequences of sheaves on S which contain both of the \mathcal{L}_i , as well as other intermediate sheaves. Taking cohomology will then give the desired result. The intermediate sheaves will depend on the particular example, and we treat each cases separately.

By what we have done so far, the problem is reduced to working with Galois modules. It only depends on the fields K_i , and by taking a suitably large extension L/F containing the normal closure of each K_i we can work with finite $G(L/F)$ -modules. The strategy is to first find a collection of exact sequences of $G(L/F)$ -modules containing the \mathcal{L}_i - each of these sequences corresponds to an exact sequence of sheaves at the generic point. Then we fix a subgroup $I \subset G(L/F)$ which is the inertia group of some ramified prime, and take I -invariants of the modules. We require that this preserve exactness, which by Lemma 12 corresponds to exactness of the sheaves at that ramified point. Lemma 14 will give the necessary description of the modules which are our starting point.

We look at examples in the case of function fields, where $S = \mathbb{P}_{\mathbb{F}_p}^1$ and $F = \mathbb{F}_p(T)$, and in number fields, where $S = \mathrm{Spec} \mathbb{Z}$ and $F = \mathbb{Q}$. Since the problem only depends on the Galois theory of the fields in question, any example has a manifestation in both settings, though the computations vary in details, such as the computation of the unit group $\mathcal{O}_X(X)^\times$, and hence give different kinds of bounds.

2. SOME COHOMOLOGY COMPUTATIONS

We begin with some lemmas which will be needed in subsequent computations. We will use the notation $h^i(X, \mathcal{F}) = \log_l |H^i(X, \mathcal{F})|$ (when the cohomology group has exponent l this is the l -rank).

2.1. Function fields. We first let $S = \mathbb{P}_{\mathbb{F}_p}^1$ and $F = \mathbb{F}_p(T)$. Finite field extensions of $\mathbb{F}_p(T)$ correspond to curves which are finite covers of $\mathbb{P}^1(\mathbb{F}_p)$. Any such curve C corresponds to its function field $\mathbb{F}_p(C)$.

We compute the unit groups mod l of a curve C and cohomology groups of the sheaf μ_l on S .

Lemma 15. *For any curve C finite over S we have*

$$\mathcal{O}_C(C)^\times / (\mathcal{O}_C(C)^\times)^l = \begin{cases} \mathbb{Z}/l\mathbb{Z} & \text{if } \mu_l \in \mathbb{F}_p \\ 1 & \text{if } \mu_l \notin \mathbb{F}_p. \end{cases}$$

Proof. Follows since $\mathcal{O}_C(C)^\times = \mathbb{F}_p^\times \cong \mathbb{Z}/(p-1)\mathbb{Z}$ and $\mu_l \in \mathbb{F}_p$ is equivalent to $l \mid p-1$. \square

Lemma 16. *For the sheaf μ_l on S we have*

$$H^0(\mu_l), H^1(\mu_l), H^2(\mu_l) = \begin{cases} \mathbb{Z}/l\mathbb{Z}, \mathbb{Z}/l\mathbb{Z}, \mathbb{Z}/l\mathbb{Z} & \text{if } \mu_l \in \mathbb{F}_p \\ 0, 0, \mathbb{Z}/l\mathbb{Z} & \text{if } \mu_l \notin \mathbb{F}_p. \end{cases}$$

Proof. We take cohomology of the Kummer sequence on \mathbb{P}^1 to get

$$\begin{array}{ccccccc}
 1 & \longrightarrow & H^0(\mu_l) & \longrightarrow & \mathbb{F}_p^\times & \longrightarrow & \mathbb{F}_p^\times \\
 & & & & \swarrow & & \\
 & & H^1(\mu_l) & \longleftarrow & \mathbb{Z} & \longrightarrow & \mathbb{Z} \\
 & & & & \swarrow & & \\
 & & H^2(\mu_l) & \longleftarrow & 0 & \longrightarrow & 0 \\
 & & & & \swarrow & & \\
 & & H^3(\mu_l) & \longleftarrow & \mathbb{Q}/\mathbb{Z} & \longrightarrow & \mathbb{Q}/\mathbb{Z}
 \end{array}$$

since $H^1(\mathbb{P}^1, \mathbb{G}_m) = \text{Pic}\mathbb{P}^1 = \mathbb{Z}$, and $H^2(\mathbb{P}^1, \mathbb{G}_m) = 0$ and $H^3(\mathbb{P}^1, \mathbb{G}_m) = \mathbb{Q}/\mathbb{Z}$ (see [4] p.109). \square

Lemma 17. *Let \mathcal{F} be a constructible sheaf on \mathbb{P}^1 with $l\mathcal{F} = 0$. Suppose that $\mathcal{F} = g_*\mathcal{F}_0$ for some sheaf \mathcal{F}_0 on $\eta = \text{Spec}F$, where $g : \eta \longrightarrow S$. Then*

$$h^2(\mathcal{F}) \leq h^1(\mathcal{F}) - h^0(\mathcal{F}) + \text{rk}_l \mathcal{F}_{\bar{\eta}}.$$

Proof. By [5] Theorem 2.13 p. 174 we have the Euler characteristic $\chi(\mathcal{F}) = h^0(\mathcal{F}) - h^1(\mathcal{F}) + h^2(\mathcal{F}) - h^3(\mathcal{F}) = 0$. Also from [5] Corollary 3 p.177 we have for any such sheaf \mathcal{F} that $H^3(\mathcal{F}) \cong H^0(\mathcal{F}^D)$ where \mathcal{F}^D is the dual sheaf defined by $\mathcal{F}^D(W) = \text{Hom}(\mathcal{F}|_W, \mu_l|_W)$. So

$$\begin{aligned}
 H^0(\mathcal{F}^D) &= \text{Hom}(\mathcal{F}, \mu_l) \\
 &= \text{Hom}(j_*\mathcal{F}_0, j_*\mu_l) \\
 &= \text{Hom}(j^*j_*\mathcal{F}_0, \mu_l) \\
 &= \text{Hom}(\mathcal{F}_0, \mu_l) \\
 &= \text{Hom}_{G_F}(\mathcal{F}_{\bar{\eta}}, \mu_l) \\
 &\subset \text{Hom}_{\mathbb{Z}/l\mathbb{Z}}(\mathcal{F}_{\bar{\eta}}, \mu_l).
 \end{aligned}$$

and the l -rank of this last set is bounded by $\text{rk}_l \mathcal{F}_{\bar{\eta}}$. Thus $h^3(\mathcal{F}) \leq \text{rk}_l \mathcal{F}_{\bar{\eta}}$ and the result follows. \square

2.2. Number fields. Now let $S = \text{Spec}\mathbb{Z}[1/l]$ and $F = \mathbb{Q}$. We have the following lemmas analogous to the function field case.

Lemma 18. *Let K be a number field and let $s = r_1 + r_2 - 1$ where r_1 and r_2 are the number of real and complex embeddings. Let u be the number of primes in K above l . Let $t = 1$ if l divides the order of μ_K and 0 otherwise. Then*

$$\mathcal{O}_K^\times[1/l] / (\mathcal{O}_K^\times[1/l])^l = (\mathbb{Z}/l\mathbb{Z})^{s+u+t}.$$

Proof. This follows from the assumption and since $\mathcal{O}_K^\times[1/l] \cong \mu_K \oplus \mathbb{Z}^{s+u}$. \square

Lemma 19. *For the sheaf μ_l on S we have*

$$H^0(\mu_l), H^1(\mu_l), H^2(\mu_l) = \begin{cases} \mathbb{Z}/2\mathbb{Z}, (\mathbb{Z}/2\mathbb{Z})^2, \mathbb{Z}/2\mathbb{Z} & \text{if } l = 2 \\ 0, \mathbb{Z}/l\mathbb{Z}, 0 & \text{if } l \neq 2 \end{cases}$$

Proof. Take cohomology of the Kummer sequence on $\text{Spec}\mathbb{Z}[1/l]$:

$$\begin{array}{ccccccc} 1 & \longrightarrow & H^0(\mu_l) & \longrightarrow & \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z} & \longrightarrow & \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z} \\ & & & & \searrow & & \searrow \\ & & H^1(\mu_l) & \longleftarrow & 0 & \longrightarrow & 0 \\ & & & & \searrow & & \searrow \\ & & H^2(\mu_l) & \longleftarrow & \mathbb{Z}/2\mathbb{Z} & \longrightarrow & \mathbb{Z}/2\mathbb{Z} \\ & & & & \searrow & & \searrow \\ & & H^3(\mu_l) & \longleftarrow & 0 & \longrightarrow & 0 \end{array}$$

We have used that $H^1(\text{Spec}\mathbb{Z}[1/l], \mathbb{G}_m) = \text{Pic}(\text{Spec}\mathbb{Z}[1/l]) = Cl_{\{\emptyset\}}(\mathbb{Q}) = 0$. Also $H^2(\text{Spec}\mathbb{Z}[1/l], \mathbb{G}_m) = \mathbb{Z}/2\mathbb{Z}$ and $H^3(\text{Spec}\mathbb{Z}[1/l], \mathbb{G}_m) = 0$ (see [4], p.109). \square

Lemma 20. *Let \mathcal{F} be a constructible sheaf on $S = \text{Spec}\mathbb{Z}[1/l]$ with $l\mathcal{F} = 0$. Suppose that $\mathcal{F} = g_*\mathcal{F}_0$ for some sheaf \mathcal{F}_0 on $\eta = \text{Spec}F$, where $g : \eta \rightarrow S$. Furthermore assume that $\mathcal{F}(\mathbb{R}) = 0$ if $l \neq 2$ and \mathcal{F} trivializes on \mathbb{R} if $l = 2$. Then*

$$h^2(\mathcal{F}) \leq h^1(\mathcal{F}) - h^0(\mathcal{F}).$$

Proof. By [5] Theorem 2.13, p.174 we have the Euler characteristic $\chi(\mathcal{F}) = h^0(\mathcal{F}) - h^1(\mathcal{F}) + h^2(\mathcal{F}) - h^3(\mathcal{F})$ is equal to

$$\chi(\mathcal{F}) = \text{rk}_l \mathcal{F}(\mathbb{R}) - \text{rk}_l H_T^0(G(\mathbb{C}/\mathbb{R}), \mathcal{F}) - \text{rk}_l \mathcal{F}(L').$$

where R is a finite set of points in S such that \mathcal{F} is locally constant on $S \setminus R$ and L' is the maximal subfield of $\overline{\mathbb{Q}}$ which is ramified only at R . So we can choose R such that $L \subset L'$ where L is some minimal number field on which \mathcal{F} trivializes. Then $\mathcal{F}(L') = \mathcal{F}_{\overline{\eta}}$. Finally H_T^0 is the Tate cohomology group and if $l = 2$ and \mathcal{F} trivializes on \mathbb{R} we get $N_{G(\mathbb{C}/\mathbb{R})} \mathcal{F}_{\text{Spec}\mathbb{R}}^{G(\mathbb{C}/\mathbb{R})} = 0$, so $H_T^0(G(\mathbb{C}/\mathbb{R}), \mathcal{F}) = 0$. Thus in both cases $\chi = -\text{rk}_l \mathcal{F}_{\overline{\eta}}$. So

$$h^2(\mathcal{F}) = h^1(\mathcal{F}) - h^0(\mathcal{F}) + h^3(\mathcal{F}) - \text{rk}_l \mathcal{F}_{\overline{\eta}}.$$

As in the proof of Lemma 17 $H^3(\mathcal{F}) \cong H^0(\mathcal{F}^D)$ and $\text{rk}_l H^0(\mathcal{F}^D) \leq \text{rk}_l \mathcal{F}_{\overline{\eta}}$. \square

3. EXACT SEQUENCES OF SHEAVES

We now consider two schemes $\pi_i : X_i \rightarrow S$ for $i = 1, 2$ of the form described in Section 1, with generic points $\text{Spec}K_i$. Let L/F be a Galois extension containing the K_i and let $G = \text{Gal}(L/F)$. We let $L' = L(\mu_l)$ and assume K_2 is disjoint from $F(\mu_l)$. Let $B = \text{Gal}(F(\mu_l)/F)$. Then either $B = (\mathbb{Z}/l\mathbb{Z})^\times$ or $B = 1$. Let $G' = \text{Gal}(L'/F) = G \times B$. Note $(|B|, l) = 1$.

Let $\mathcal{L}_i = (\pi_i)_* \mu_l$. Let $M = \mathcal{L}_{1, \overline{\eta}}$ and $N = \mathcal{L}_{2, \overline{\eta}}$. These are G_F -modules where the action factors through $G \times B$ and B acts by multiplication. Since the B action will

have no bearing on the results in this section we disregard it for now and simply talk about G -modules. The B action will become important in the next section when we are computing cohomology.

In each case we apply Lemma 14 to give a description of M and N and then compute a series of exact sequences relating them.

Remark 21. Since all modules will be G -modules, by Lemma 11 they will represent sheaves on U , the set of points where neither π_1 or π_2 ramify. We will then apply the functor j_* to obtain sheaves on S (noting by 13 that this recovers the sheaves \mathcal{L}_i). To check exactness of a sequence of sheaves is equivalent to checking exactness of stalks at each geometric point. Thus by Lemma 12 we need to check the original sequences of modules remain exact after taking I_z -invariants by the inertia group I_z of each ramified point z .

3.1. The case of S_3 . Let $l = 3$. Let L/F be a Galois extension with $\text{Gal}(L/F) = S_3$. Let K_1 be a non-Galois cubic subfields. Let K_2 be the unique quadratic subfield (this is called the quadratic resolvent of K_1).

Let $S_3 = \langle \sigma, \tau \mid \sigma^2 = \tau^3 = 1, \sigma\tau\sigma = \tau^{-1} \rangle$ be the usual presentation. Then as S_3 -modules

$$M = \mathbb{Z}/3\mathbb{Z} \langle 1, \tau, \tau^2 \rangle$$

where τ^i represents the coset modulo the subgroup $\langle \sigma \rangle \subset S_3$ and similarly

$$N = \mathbb{Z}/3\mathbb{Z} \langle 1, \sigma \rangle$$

where σ^i represents the coset modulo the subgroup $\langle \tau \rangle \subset S_3$.

We now write down a series of exact sequences relating these modules. Let $N' = \mathbb{Z}/3\mathbb{Z} \langle 1 - \sigma \rangle$. Let T be the module representing the sheaf μ_3 . It is one dimensional with trivial G action. Then we have

$$(3.1) \quad \begin{array}{ccccccc} 0 & \longrightarrow & \langle 1 + \tau + \tau^2 \rangle & \longrightarrow & M & \longrightarrow & M' \longrightarrow 0 \\ 0 & \longrightarrow & N' & \longrightarrow & M' & \longrightarrow & T \longrightarrow 0. \end{array}$$

The map $N' \longrightarrow M'$ is given by $1 - \sigma \longmapsto 1 - \tau$. We also have

$$(3.2) \quad N = N' \oplus \langle 1 + \sigma \rangle.$$

Note that T corresponds to the sheaf μ_3 .

This is a sequence of sheaves at the generic point. Now we apply 21. Here the assumption that L/K_2 is unramified implies that the inertia group at any point has order 2. It is a fact that $H^1(G, M) = 0$ whenever the order of G is coprime to M . Since all of the above modules have order a power of 3 we obtain exact sequences of sheaves on S . Note also that T will be extended to μ_3 on S .

3.2. The case of S_4 and A_4 . Let L/F be a Galois extensions with $\text{Gal}(L/F) = S_4$. Let K_1 be a non-Galois quartic subfield. There is a unique subgroup of S_4 isomorphic to the Klein-four group V_4 which is furthermore normal in S_4 . Its fixed field has Galois group S_3 , hence has 3 cubic subfields. Let K_2 be one of these fields (this is called the cubic resolvent of K_1).

Then

$$\begin{aligned} M &= \mathbb{Z}/2\mathbb{Z} \langle a_1, a_2, a_3, a_4 \rangle, \\ N &= \mathbb{Z}/2\mathbb{Z} \langle b_1, b_2, b_3 \rangle. \end{aligned}$$

The action of S_4 permutes the a_i 's and factors through S_3 on the b_i 's.

Let $M_1 = \langle a_2 + a_1, a_3 + a_1, a_4 + a_1 \rangle$ and $N' = \mathbb{Z}/2\mathbb{Z} \langle a_1 + a_2, a_1 + a_3 \rangle / \langle a_1 + a_2 + a_3 + a_4 \rangle$. Let T be the trivial module of dimension 1. Then we have

$$(3.3) \quad \begin{array}{ccccccc} 0 & \longrightarrow & M_1 & \longrightarrow & M & \longrightarrow & T \longrightarrow 0 \\ 0 & \longrightarrow & T & \longrightarrow & M_1 & \longrightarrow & N' \longrightarrow 0 \end{array}$$

Furthermore we have

$$(3.4) \quad N = N' \oplus T$$

where N' lies in N as $a_1 + a_2 \mapsto b_3$ and $a_1 + a_3 \mapsto b_2$.

Now we apply Remark 21. We assume that the inertia group I of any point is disjoint from V_4 . Then the sequences of I -invariants remain exact. This is clear for any subgroup of order 3. It is straightforward to check that for any subgroup I generated by a transposition of S_4 , $\dim_{\mathbb{F}_2} M^I = 3$. Then $\dim_{\mathbb{F}_2} M_1^I \geq 2$ and since $M_1^I \neq M_1$ it must be that $\dim_{\mathbb{F}_2} M_1^I = 2$ so the top sequence remains exact. It is also easy to check that $\dim_{\mathbb{F}_2} N'^I = 1$ so the middle sequence remains exact. This also implies that exactness is preserved under $I = S_3 \subset S_4$ since then I is generated by a transposition and a 3-cycle.

In the case of $\text{Gal}(L/F) = A_4$ there is also a unique normal subgroup isomorphic to V_4 . Let K_2 be its fixed field which is Galois with group C_3 . Again K_1 is one of the quartic subfields. We also assume that I is disjoint from V_4 . Then the sequences relating M and N are exactly the same as above and the argument for the I -invariants is similar.

3.3. The case of D_{2l} . We can generalize the above method to the following case. Let l be an odd prime. Let $D_{2l} = \langle \sigma, \tau \mid \sigma^2 = \tau^l = 1, \sigma\tau\sigma = \tau^{-1} \rangle$. Let L/F be a galois extension with $\text{Gal}(L/F) = D_{2l}$. Let K_1 be a non-Galois subextension of degree l . Let K_2 be the unique quadratic subfield of L .

Then

$$\begin{aligned} M &= \mathbb{Z}/l\mathbb{Z} \langle 1, \tau, \dots, \tau^{l-1} \rangle \\ N &= \mathbb{Z}/l\mathbb{Z} \langle 1, \sigma \rangle \end{aligned}$$

where the basis elements are assumed to represent the cosets of D_{2l} by the appropriate subgroup.

We want to write down a series of exact sequences relating these two modules. We start by computing the Jordan-Holder decomposition of M . For each $k = 0, \dots, l-1$ let

$$M_k = \left\{ \sum_{i=0}^{l-1} a(i) \tau^i \mid a(x) \text{ a degree } k \text{ polynomial} \right\}$$

where the coefficients are taken in $\mathbb{Z}/l\mathbb{Z}$. Then it is not hard to see that this is a filtration of G -modules, with

$$M_k/M_{k-1} = \begin{cases} \mathbb{Z}/l\mathbb{Z} & k \text{ even} \\ N' & k \text{ odd} \end{cases}$$

where $N' = \mathbb{Z}/l\mathbb{Z} \langle 1 - \sigma \rangle$ as in the S_3 example. Furthermore $M_0 = \mathbb{Z}/l\mathbb{Z}$ and $M_{l-1} = M$. This can be restated as the series of exact sequences

$$(3.5) \quad \begin{array}{ccccccc} 0 & \longrightarrow & M_{l-2} & \longrightarrow & M_{l-1} & \longrightarrow & \mathbb{Z}/l\mathbb{Z} \longrightarrow 0 \\ 0 & \longrightarrow & M_{l-3} & \longrightarrow & M_{l-2} & \longrightarrow & N' \longrightarrow 0 \\ & & & & \vdots & & \\ 0 & \longrightarrow & M_0 & \longrightarrow & M_1 & \longrightarrow & N' \longrightarrow 0 \\ 0 & \longrightarrow & 0 & \longrightarrow & M_0 & \longrightarrow & \mathbb{Z}/l\mathbb{Z} \longrightarrow 0 \end{array}$$

We also have

$$(3.6) \quad N = N' \oplus \mathbb{Z}/l\mathbb{Z}.$$

Now we apply Remark 21. Again using the fact that $H^1(G, M) = 0$ whenever the order of G is coprime to M for any G -module M , we see that taking invariants by any subgroup of G of order coprime to l preserves exactness.

3.4. The case of $\mathbb{Z}/l\mathbb{Z} \rtimes \mathbb{Z}/r\mathbb{Z}$. Let L/F be a Galois extensions with $G = \mathbb{Z}/l\mathbb{Z} \rtimes \mathbb{Z}/r\mathbb{Z}$. We use the presentation of $G = \{ \sigma, \tau \mid \sigma^r = \tau^l = 1, \sigma\tau\sigma^{-1} = \tau^\zeta \}$ where $\zeta \in (\mathbb{Z}/l\mathbb{Z})^\times$ denotes an r th root of unity (since $r \mid l-1$). We assume $l \equiv 1 \pmod{r}$. Let K_1 be one of the conjugate degree l subfields and let K_2 be the degree r Galois subfield.

Then $M = \mathbb{Z}/l\mathbb{Z} \langle 1, \tau, \dots, \tau^{l-1} \rangle$ and $N = \mathbb{Z}/l\mathbb{Z} \langle 1, \sigma, \dots, \sigma^{r-1} \rangle$.

We can describe a filtration of N as follows. For each $k = 0, \dots, r-1$ let

$$N_k = \left\{ \sum_{i=0}^{r-1} a(\zeta^i) \sigma^i \mid a(x) \text{ a degree } k \text{ polynomial} \right\}$$

where the coefficients are taken in $\mathbb{Z}/l\mathbb{Z}$. Then it is not hard to see that this is a filtration of G -modules (with τ acting trivially), with

$$N_k/N_{k-1} = R_k$$

where $R_k = \mathbb{Z}/l\mathbb{Z}$ on which σ acts as multiplication by ζ^{-k} . Furthermore $N_0 = \mathbb{Z}/l\mathbb{Z}$ and $N_{r-1} = N$. Since $\langle \sigma \rangle$ is cyclic and has order coprime to l , the representation N decomposes as a direct sum of 1 dimensional representations. This implies that

$$(3.7) \quad N = \bigoplus_{k=0}^{r-1} R_k.$$

Similarly we describe a filtration of M . For each $k = 0, \dots, l-1$ let

$$M_k = \left\{ \sum_{i=0}^{l-1} a(i) \tau^i \mid a(x) \text{ a degree } k \text{ polynomial} \right\}.$$

This is a filtration of G -modules with the same factors

$$M_k/M_{k-1} = R_{k(\text{mod } l)}.$$

Note however that here each R_k appears $n = (l-1)/r$ times except R_0 which appears $n+1$ times. Here $M_{l-1} = M$. This can be restated as the series of exact sequences

$$(3.8) \quad \begin{array}{ccccccc} 0 & \longrightarrow & M_{l-2} & \longrightarrow & M_{l-1} & \longrightarrow & \mathbb{Z}/l\mathbb{Z} \longrightarrow 0 \\ 0 & \longrightarrow & M_{l-3} & \longrightarrow & M_{l-2} & \longrightarrow & R_{l-1} \longrightarrow 0 \\ & & & & \vdots & & \\ 0 & \longrightarrow & M_0 & \longrightarrow & M_1 & \longrightarrow & R_1 \longrightarrow 0 \\ 0 & \longrightarrow & 0 & \longrightarrow & M_0 & \longrightarrow & \mathbb{Z}/l\mathbb{Z} \longrightarrow 0 \end{array}$$

Now we apply Remark 21. If we assume that L has inertia group coprime to l for every prime ramified in L/\mathbb{Q} then the sequences of inertial invariants remains exact.

4. TAKING COHOMOLOGY IN FUNCTION FIELDS

In this section we take cohomology of the sequences computed in Section 3 to obtain relations between $H^1(\mathcal{L}_1)$ and $H^2(\mathcal{L}_2)$ which by Lemmas 10 and 15 give a relation between $\text{Pic}C_1$ and $\text{Pic}C_2$.

4.1. The case of S_3 .

Lemma 22. *If $G = S_3$ and $\mu_3 \in \mathbb{F}_p$ then*

$$\text{rk}_3 \text{Pic}C_2 - 2 \leq \text{rk}_3 \text{Pic}C_1 \leq \text{rk}_3 \text{Pic}C_2.$$

Proof. By Lemma 16 we have $H^i(\mu_3) = \mathbb{Z}/3\mathbb{Z}$ for $i = 0, 1, 2$. Furthermore $H^0(M) = H^0(N) = \mathbb{Z}/3\mathbb{Z}$ and $H^0(M') = H^0(N') = 0$.

Now we take cohomology of the sequences 3.1. Using the same notation to denote the sheaves extended to \mathbb{P}^1 the first sequence gives

$$\begin{array}{ccccccc}
0 & \longrightarrow & \mathbb{Z}/3\mathbb{Z} & \longrightarrow & \mathbb{Z}/3\mathbb{Z} & \longrightarrow & 0 \\
& & & & \swarrow & & \\
& & \mathbb{Z}/3\mathbb{Z} & \xleftarrow{\quad} & H^1(M) & \longrightarrow & H^1(M') \\
& & & & \swarrow & & \\
& & \mathbb{Z}/3\mathbb{Z} & \xleftarrow{\quad} & H^2(M) & \longrightarrow & H^2(M')
\end{array}$$

and the second sequence gives

$$\begin{array}{ccccccc}
0 & \longrightarrow & 0 & \longrightarrow & 0 & \longrightarrow & \mathbb{Z}/3\mathbb{Z} \\
& & & & \swarrow & & \\
& & H^1(N') & \xleftarrow{\quad} & H^1(M') & \longrightarrow & \mathbb{Z}/3\mathbb{Z} \\
& & & & \swarrow & & \\
& & H^2(N') & \xleftarrow{\quad} & H^2(M') & \longrightarrow & \mathbb{Z}/3\mathbb{Z}.
\end{array}$$

Thus we get the equalities

$$1 + h^1(M') = s + h^1(M),$$

$$1 + h^1(M') = t + h^1(N')$$

where $s \leq 1$ and $t \leq 1$. From 3.2 we also get

$$1 + h^1(N') = h^1(N).$$

Putting these together gives the bounds

$$h^1(N) - 2 \leq h^1(M) \leq h^1(N).$$

Thus using Lemma 10 we have $\text{rk}_3\text{Pic}(C_1) = h^1(M) - 1$ from the beginning, and similarly for C_2 . \square

Now suppose $\mu_3 \notin \mathbb{F}_p$. Then by Lemma 16 we have $H^i(\mu_3) = 0$ for $i = 0, 1$ and $H^2(\mu_3) = \mathbb{Z}/3\mathbb{Z}$. Furthermore $H^0(M) = H^0(N) = H^0(M') = H^0(N') = 0$ since these are the G -invariants of the stalk and in this case G has an element which acts as multiplication by 2, which fixes no element of any of the stalks. A similar computation to the above gives:

Lemma 23. *If $G = S_3$ and $\mu_3 \notin \mathbb{F}_p$ then*

$$\text{rk}_3\text{Pic}C_2 - 1 \leq \text{rk}_3\text{Pic}C_1 \leq \text{rk}_3\text{Pic}C_2.$$

4.2. The case of S_4 and A_4 .

Lemma 24. *If $G = S_4$ or A_4 then*

$$\text{rk}_2\text{Pic}C_2 - 2 \leq \text{rk}_2\text{Pic}C_1 \leq \text{rk}_2\text{Pic}C_2.$$

Proof. By Lemma 16 we have $H^i(\mu_2) = \mathbb{Z}/2\mathbb{Z}$ for $i = 0, 1, 2$. Furthermore $H^0(M) = H^0(N) = \mathbb{Z}/2\mathbb{Z}$. We consider the filtration 3.2. Taking cohomology of the first sequences gives

$$\begin{array}{ccccccc} 0 & \longrightarrow & H^0(M_1) & \longrightarrow & \mathbb{Z}/2\mathbb{Z} & \longrightarrow & \mathbb{Z}/2\mathbb{Z} \\ & & & & \searrow & & \\ & & H^1(M_1) & \longleftarrow & H^1(M) & \longrightarrow & \mathbb{Z}/2\mathbb{Z} \\ & & & & \searrow & & \\ & & H^2(M_1) & \longleftarrow & H^2(M) & \longrightarrow & \mathbb{Z}/2\mathbb{Z} \end{array}$$

and the second sequence gives

$$\begin{array}{ccccccc} 0 & \longrightarrow & \mathbb{Z}/2\mathbb{Z} & \longrightarrow & H^0(M_1) & \longrightarrow & 0 \\ & & & & \searrow & & \\ & & \mathbb{Z}/2\mathbb{Z} & \longleftarrow & H^1(M_1) & \longrightarrow & H^1(N') \\ & & & & \searrow & & \\ & & \mathbb{Z}/2\mathbb{Z} & \longleftarrow & H^2(M_1) & \longrightarrow & H^2(N') \end{array}$$

These sequences give the equalities

$$h^1(M) + 1 = h^1(M_1) + s$$

$$h^1(M_1) + t = h^1(N') + 1$$

where $s \leq 1, t \leq 1$. From 3.4 we also get

$$1 + h^1(N') = h^1(N).$$

Combining these gives the bounds

$$h^1(N) - 2 \leq h^1(M) \leq h^1(N).$$

□

4.3. The case of D_{2l} .

Lemma 25. *If $G = D_{2l}$ and $\mu_l \in \mathbb{F}_p$ then*

$$\mathrm{rk}_l \mathrm{Pic}(C_2) - \frac{l-1}{2} \leq \mathrm{rk}_l \mathrm{Pic}(C_1) \leq \frac{l-1}{2} \mathrm{rk}_l \mathrm{Pic}(C_2) + 1.$$

Proof. Since $\mu_l \in \mathbb{F}_p$ in this case B is trivial, $L = L'$ and we are just looking at sequences of D_{2l} -modules. Then by Lemma 16 $H^i(\mu_l) = \mathbb{Z}/l\mathbb{Z}$ for $i = 0, 1, 2$. Furthermore $H^0(N) = H^0(M) = \mathbb{Z}/l\mathbb{Z}$. This implies that $H^0(M_k) = \mathbb{Z}/l\mathbb{Z}$ for all k and $H^0(N') = 0$.

We consider the filtration 3.3. Taking cohomology of the first sequence gives

$$\begin{array}{ccccccc}
0 & \longrightarrow & \mathbb{Z}/l\mathbb{Z} & \longrightarrow & \mathbb{Z}/l\mathbb{Z} & \longrightarrow & \mathbb{Z}/l\mathbb{Z} \\
& & & & \swarrow & & \\
& & H^1(M_{l-2}) & \xleftarrow{\quad} & H^1(M_{l-1}) & \longrightarrow & \mathbb{Z}/l\mathbb{Z} \\
& & & & \swarrow & & \\
& & H^2(M_{l-2}) & \xleftarrow{\quad} & H^2(M_{l-1}) & \longrightarrow & \mathbb{Z}/l\mathbb{Z}.
\end{array}$$

Taking cohomology of the second sequence gives

$$\begin{array}{ccccccc}
0 & \longrightarrow & \mathbb{Z}/l\mathbb{Z} & \longrightarrow & \mathbb{Z}/l\mathbb{Z} & \longrightarrow & 0 \\
& & & & \swarrow & & \\
& & H^1(M_{l-3}) & \xleftarrow{\quad} & H^1(M_{l-2}) & \longrightarrow & H^1(N') \\
& & & & \swarrow & & \\
& & H^2(M_{l-3}) & \xleftarrow{\quad} & H^2(M_{l-2}) & \longrightarrow & H^2(N').
\end{array}$$

And so on. The $k = 1$ sequence gives

$$\begin{array}{ccccccc}
0 & \longrightarrow & \mathbb{Z}/l\mathbb{Z} & \longrightarrow & \mathbb{Z}/l\mathbb{Z} & \longrightarrow & 0 \\
& & & & \swarrow & & \\
& & \mathbb{Z}/l\mathbb{Z} & \xleftarrow{\quad} & H^1(M_1) & \longrightarrow & H^1(N') \\
& & & & \swarrow & & \\
& & \mathbb{Z}/l\mathbb{Z} & \xleftarrow{\quad} & H^2(M_1) & \longrightarrow & H^2(N').
\end{array}$$

Hence we get the equalities

$$\begin{aligned}
h^1(M_k) + 1 &= h^1(M_{k-1}) + s_k \text{ if } k \text{ is even,} \\
h^1(M_k) &= t_k + h^1(M_{k-1}) \text{ if } k \text{ is odd,} \\
v + h^1(M_1) &= h^1(N') + 1
\end{aligned}$$

where $s_k \leq 1$, $t_k \leq h^1(N')$ and $v \leq 1$. By 3.6 we also have

$$h^1(N) = h^1(N') + 1.$$

Putting all of these together gives the bounds

$$h^1(N) - \frac{l-1}{2} - 1 \leq h^1(M) \leq \frac{l-1}{2} (h^1(N) - 1) + 1.$$

□

Now suppose $\mu_l \notin \mathbb{F}_p$. In this case B is non-trivial. Then by Lemma 16 $H^i(\mu_l) = 0$ for $i = 0, 1$ and $H^2(\mu_l) = \mathbb{Z}/l\mathbb{Z}$. Furthermore $H^0(M) = H^0(N) = 0$. This implies that $H^0(M_k) = 0$ for all k and $H^0(N') = 0$. A similar computation gives:

Lemma 26. *If $G = D_{2l}$ and $\mu_l \notin \mathbb{F}_p$ then*

$$\mathrm{rk}_l \mathrm{Pic}(C_2) - 1 \leq \mathrm{rk}_l \mathrm{Pic}(C_1) \leq \frac{l-1}{2} \mathrm{rk}_l \mathrm{Pic}(C_2).$$

4.4. The case of $\mathbb{Z}/l\mathbb{Z} \rtimes \mathbb{Z}/r\mathbb{Z}$.

Lemma 27. *If $G = \mathbb{Z}/l\mathbb{Z} \rtimes \mathbb{Z}/r\mathbb{Z}$ and $\mu_l \in \mathbb{F}_p$ then*

$$\frac{1}{r-1}(\mathrm{rk}_l \mathrm{Pic} C_2) - \frac{l-3}{r-1} - 2 \leq \mathrm{rk}_l \mathrm{Pic} C_1 \leq \frac{(l-1)}{r}(\mathrm{rk}_l \mathrm{Pic} C_2 + 1).$$

Proof. Suppose $\mu_l \in \mathbb{F}_p$. Then by Lemma 16 we have $H^i(\mu_l) = \mathbb{Z}/l\mathbb{Z}$ for $i = 0, 1, 2$. Furthermore $H^0(M) = H^0(N) = \mathbb{Z}/l\mathbb{Z}$. This implies that $H^0(M_k) = \mathbb{Z}/l\mathbb{Z}$ for all k and $H^0(N_k) = \mathbb{Z}/l\mathbb{Z}$ for all k .

By 3.7 and since cohomology commutes with direct sums we get

$$h^1(N) = \sum_{k=0}^{r-1} h^1(R_k).$$

Now we focus on the filtration 3.4. Taking cohomology of the k th sequence when $k \neq 1$ and $R_k \neq \mathbb{Z}/l\mathbb{Z}$ gives:

$$\begin{array}{ccccccc} 0 & \longrightarrow & \mathbb{Z}/l\mathbb{Z} & \longrightarrow & \mathbb{Z}/l\mathbb{Z} & \longrightarrow & 0 \\ & & & & \searrow & & \\ & & H^1(M_{k-1}) & \xleftarrow{\quad} & H^1(M_k) & \longrightarrow & H^1(R_k) \\ & & & & \searrow & & \\ & & H^2(M_{k-1}) & \xleftarrow{\quad} & H^2(M_k) & \longrightarrow & H^2(R_k) \end{array}$$

and when $k = 1$ gives:

$$\begin{array}{ccccccc} 0 & \longrightarrow & \mathbb{Z}/l\mathbb{Z} & \longrightarrow & \mathbb{Z}/l\mathbb{Z} & \longrightarrow & 0 \\ & & & & \searrow & & \\ & & \mathbb{Z}/l\mathbb{Z} & \xleftarrow{\quad} & H^1(M_1) & \longrightarrow & H^1(R_1) \\ & & & & \searrow & & \\ & & \mathbb{Z}/l\mathbb{Z} & \xleftarrow{\quad} & H^2(M_1) & \longrightarrow & H^2(R_1) \end{array}$$

and when $R_k = \mathbb{Z}/l\mathbb{Z}$ gives:

$$\begin{array}{ccccccc} 0 & \longrightarrow & \mathbb{Z}/l\mathbb{Z} & \longrightarrow & \mathbb{Z}/l\mathbb{Z} & \longrightarrow & \mathbb{Z}/l\mathbb{Z} \\ & & & & \searrow & & \\ & & H^1(M_{k-1}) & \xleftarrow{\quad} & H^1(M_k) & \longrightarrow & \mathbb{Z}/l\mathbb{Z} \\ & & & & \searrow & & \\ & & H^2(M_{k-1}) & \xleftarrow{\quad} & H^2(M_k) & \longrightarrow & \mathbb{Z}/l\mathbb{Z}. \end{array}$$

Now by Lemma 17

$$\begin{aligned} h^2(M_k) &\leq h^1(M_k) - h^0(M_k) + \mathrm{rk}_l M_k \\ &= h^1(M_k) + k \end{aligned}$$

for all k . Hence for all $R_k \neq \mathbb{Z}/l\mathbb{Z}$ and $k \neq 1$

$$s_k + h^1(M_k) = h^1(R_k) + h^1(M_{k-1})$$

where $s_k \leq h^1(M_{k-1}) + (k-1)$. And for $k=1$

$$s_1 + h^1(M_1) = h^1(R_1) + 1$$

where $s_1 \leq 1$. And for all $R_k = \mathbb{Z}/l\mathbb{Z}$

$$h^1(M_k) = t_k + h^1(M_{k-1}) - 1$$

where $t_k \leq 1$. Combining this gives the upper bound

$$h^1(M) \leq \frac{l-1}{r} \sum_{k=1}^{l-1} h^1(R_k) + 1.$$

To get a lower bound we note that for $R_k \neq \mathbb{Z}/l\mathbb{Z}$ and $k \neq 1$ we have both

$$\begin{aligned} h^1(R_k) - (k-1) &\leq h^1(M_k), \\ h^1(M_{k-1}) &\leq h^1(M_k). \end{aligned}$$

and for $R_k = \mathbb{Z}/l\mathbb{Z}$ we have $h^1(M_{k-1}) - 1 \leq h^1(M_k)$ so

$$h^1(M) \geq \max_k \{h^1(R_k) - (k-1)\} - 1 \geq \frac{1}{r-1} \sum_{k=1}^{r-1} h^1(R_k) - \frac{l-3}{r-1} - 1$$

(note in the above max k is bounded above by $l-2$). Putting it all together we get

$$\frac{1}{r-1} (h^1(N) - 1) - \frac{l-3}{r-1} - 1 \leq h^1(M) \leq \frac{(l-1)}{r} (h^1(N) - 1) + 1.$$

□

Now suppose $\mu_l \notin \mathbb{F}_p$. Then by Lemma 16 we have $H^i(\mu_l) = 0$ for $i = 0, 1$ and $H^2(\mu_l) = \mathbb{Z}/l\mathbb{Z}$. Furthermore $H^0(M) = H^0(N) = 0$. This implies that $H^0(M_k) = 0$ for all k and $H^0(N_k) = 0$ for all k . A similar computation gives:

Lemma 28. *If $G = \mathbb{Z}/l\mathbb{Z} \rtimes \mathbb{Z}/r\mathbb{Z}$ and $\mu_l \notin \mathbb{F}_p$ then*

$$\frac{1}{r-1} \text{rk}_l \text{Pic} C_2 - \frac{l-2}{r-1} \leq \text{rk}_l \text{Pic} C_1 \leq \frac{(l-1)}{r} \text{rk}_l \text{Pic} C_2.$$

5. TAKING COHOMOLOGY IN NUMBER FIELDS

The same approach works in the case of number fields, with the curves replaced by $\text{Spec} \mathcal{O}_K[1/l]$ where \mathcal{O}_K is the ring of integers of the number field K . We adjoin the element $1/l$ to ensure that the Kummer sequence for the prime l is exact.

Remark 29. Let S be a finite set of primes in K and let $Cl_S(K)$ be the class group of K away from the primes in S . The relation with the usual class group is as follows. There is a surjective morphism

$$\phi : Cl(K) \longrightarrow Cl_S(K)$$

which sends primes in S to the trivial class. Thus $\text{rk}_l Cl_S(K) + \text{rk}_l \ker \phi = \text{rk}_l Cl(K)$. An element of $Cl(K)$ is in the kernel if it has a representative supported only on primes

in S . Thus $\text{rk}_l \ker \phi \leq l^{|S|-1}$, with the -1 in the exponent coming from the fact that there is always at least one relation between the full set of primes in K lying above any prime.

In this section we will let S consist of the set of primes above l . The above remark is important because $\text{Pic}(\text{Spec } \mathcal{O}_K[1/l]) = Cl_S(K)$. We let $s_i = r_1(K_i) + r_2(K_i) - 1$ and u_i denote the number of primes in K_i above l . Finally let $t_i = s_i + u_i$.

The computations are very similar to the previous section, and we simply state the exact sequences and the final result in each case, omitting computations.

5.1. The case of S_3 .

Lemma 30. *If $G = S_3$ and $K_2 \neq \mathbb{Q}(\zeta_3)$ then*

$$\text{rk}_3 Cl_S(K_2) - 1 + (t_2 - t_1) \leq \text{rk}_3 Cl_S(K_1)$$

and

$$\text{rk}_3 Cl_S(K_1) \leq \text{rk}_3 Cl_S(K_2) + (t_2 - t_1).$$

Proof. By Lemma 19 we have $H^1(\mu_3) = \mathbb{Z}/3\mathbb{Z}$ and $H^i(\mu_3) = 0$ for $i = 0, 2$. Furthermore $H^0(M) = H^0(N) = 0$ and $H^0(M') = H^0(N')$. We have:

We consider the filtration 3.1. Taking cohomology of the first sequence gives

$$\begin{array}{ccccccc} 0 & \longrightarrow & 0 & \longrightarrow & 0 & \longrightarrow & H^0(M') \\ & & & & & \searrow & \\ & & \mathbb{Z}/3\mathbb{Z} & \xleftarrow{\quad} & H^1(M) & \longrightarrow & H^1(M') \\ & & & & & \searrow & \\ & & 0 & \xleftarrow{\quad} & H^2(M) & \longrightarrow & H^2(M') \end{array}$$

and the second sequence gives

$$\begin{array}{ccccccc} 0 & \longrightarrow & H^0(N') & \longrightarrow & H^0(M') & \longrightarrow & 0 \\ & & & & & \searrow & \\ & & H^1(N') & \xleftarrow{\quad} & H^1(M') & \longrightarrow & \mathbb{Z}/3\mathbb{Z} \\ & & & & & \searrow & \\ & & H^2(N') & \xleftarrow{\quad} & H^2(M') & \longrightarrow & 0. \end{array}$$

Putting these together gives the desired result. □

5.2. The case of S_4 and A_4 .

Lemma 31. *Let $G = S_4$ or A_4 . Then*

$$\text{rk}_2 Cl_S(K_2) + (t_2 - t_1) - 2 \leq \text{rk}_2 Cl_S(K_1)$$

and

$$\text{rk}_2 Cl_S(K_1) \leq \text{rk}_2 Cl_S(K_2) + (t_2 - t_1) + 1.$$

Proof. We consider the filtration 3.2. By Lemma 19 we have $H^1(\mu_2) = (\mathbb{Z}/2\mathbb{Z})^2$ and $H^i(\mu_2) = \mathbb{Z}/2\mathbb{Z}$ for $i = 0, 2$. Furthermore $H^0(M) = H^0(N) = \mathbb{Z}/2\mathbb{Z}$. Taking cohomology of the first sequence gives

$$\begin{array}{ccccccc} 0 & \longrightarrow & H^0(M_1) & \longrightarrow & \mathbb{Z}/2\mathbb{Z} & \longrightarrow & \mathbb{Z}/2\mathbb{Z} \\ & & & & \searrow & & \\ & & H^1(M_1) & \longleftarrow & H^1(M) & \longrightarrow & (\mathbb{Z}/2\mathbb{Z})^2 \\ & & & & \searrow & & \\ & & H^2(M_1) & \longleftarrow & H^2(M) & \longrightarrow & \mathbb{Z}/2\mathbb{Z} \end{array}$$

and the second sequence gives

$$\begin{array}{ccccccc} 0 & \longrightarrow & \mathbb{Z}/2\mathbb{Z} & \longrightarrow & H^0(M_1) & \longrightarrow & 0 \\ & & & & \searrow & & \\ & & (\mathbb{Z}/2\mathbb{Z})^2 & \longleftarrow & H^1(M_1) & \longrightarrow & H^1(N') \\ & & & & \searrow & & \\ & & \mathbb{Z}/2\mathbb{Z} & \longleftarrow & H^2(M_1) & \longrightarrow & H^2(N'). \end{array}$$

From 3.4 we also get

$$2 + h^1(N') = h^1(N).$$

Putting these together gives

$$h^1(N) - 2 \leq h^1(M) \leq h^1(N) + 1$$

and the result follows. \square

Consider the case of A_4 . In this case $-3 \leq u_2 - u_1 \leq 2$. Furthermore we can compute $s_1 - s_2$ as follows. If L is totally real then $\text{sig}(K_1) = (4, 0)$ and $\text{sig}(K_2) = (3, 0)$ so $s_1 - s_2 = 1$ so

$$\text{rk}_2 Cl_S(K_2) - 3 + u_2 - u_1 \leq \text{rk}_2 Cl_S(K_1) \leq \text{rk}_2 Cl_S(K_2) + u_2 - u_1,$$

and if L is totally complex then $\text{sig}(K_1) = (0, 2)$ and $\text{sig}(K_2) = (3, 0)$ (since in this case K_2 is Galois so must be totally real) so $s_1 - s_2 = -1$ and

$$\text{rk}_2 Cl_S(K_2) - 1 + u_2 - u_1 \leq \text{rk}_2 Cl_S(K_1) \leq \text{rk}_2 Cl_S(K_2) + 2 + u_2 - u_1.$$

Combining this with Remark 29 when L is totally real gives

$$\text{rk}_2 Cl(K_2) - 10 \leq \text{rk}_2 Cl(K_1) \leq \text{rk}_2 Cl(K_2) + 10$$

and when L is totally complex gives

$$\text{rk}_2 Cl(K_2) - 8 \leq \text{rk}_2 Cl(K_1) \leq \text{rk}_2 Cl(K_2) + 12.$$

5.3. The case of D_{2l} .

Lemma 32. *If $G = D_{2l}$ and K_2 is disjoint from $\mathbb{Q}(\zeta_l)$ then*

$$\mathrm{rk}_l \mathrm{Cl}_S(K_2) + (t_2 - t_1) \leq \mathrm{rk}_l \mathrm{Cl}_S(K_1)$$

and

$$\mathrm{rk}_l \mathrm{Cl}_S(K_1) \leq \frac{l-1}{2} (\mathrm{rk}_l \mathrm{Cl}_S(K_2) + t_2 + 1) - t_1 - 1.$$

Proof. By Lemma 16 $H^1(\mu_l) = \mathbb{Z}/l\mathbb{Z}$ and $H^1(\mu_l) = 0$ for $i = 0, 2$. Furthermore $H^0(N) = H^0(M) = 0$. This implies that $H^0(M_k) = 0$ for all k and $H^0(N') = 0$.

We consider the filtration 3.3. Taking cohomology of the first sequence gives

$$\begin{array}{ccccccc} 0 & \longrightarrow & 0 & \longrightarrow & 0 & \longrightarrow & 0 \\ & & & & & \swarrow & \\ & & H^1(M_{l-2}) & \longleftarrow & H^1(M_{l-1}) & \longrightarrow & \mathbb{Z}/l\mathbb{Z} \\ & & & & & \swarrow & \\ & & H^2(M_{l-2}) & \longleftarrow & H^2(M_{l-1}) & \longrightarrow & 0. \end{array}$$

Taking cohomology of the second sequence gives

$$\begin{array}{ccccccc} 0 & \longrightarrow & 0 & \longrightarrow & 0 & \longrightarrow & 0 \\ & & & & & \swarrow & \\ & & H^1(M_{l-3}) & \longleftarrow & H^1(M_{l-2}) & \longrightarrow & H^1(N') \\ & & & & & \swarrow & \\ & & H^2(M_{l-3}) & \longleftarrow & H^2(M_{l-2}) & \longrightarrow & H^2(N'). \end{array}$$

And so on. The $k = 1$ sequence gives

$$\begin{array}{ccccccc} 0 & \longrightarrow & 0 & \longrightarrow & 0 & \longrightarrow & 0 \\ & & & & & \swarrow & \\ & & \mathbb{Z}/l\mathbb{Z} & \longleftarrow & H^1(M_1) & \longrightarrow & H^1(N') \\ & & & & & \swarrow & \\ & & 0 & \longleftarrow & H^2(M_1) & \longrightarrow & H^2(N'). \end{array}$$

By 3.6 we again have

$$h^1(N) = h^1(N') + 1.$$

Putting these together gives the desired result. \square

Compare this result with Proposition 3. Both the upper and lower bounds are worse in our case.

5.4. The case of $\mathbb{Z}/l\mathbb{Z} \rtimes \mathbb{Z}/r\mathbb{Z}$.

Lemma 33. *If $G = \mathbb{Z}/l\mathbb{Z} \rtimes \mathbb{Z}/r\mathbb{Z}$ and K_2 is disjoint from $\mathbb{Q}(\zeta_l)$ then*

$$\frac{1}{r-1} (\text{rk}_l \text{Cl}_S(K_2) + t_2) - t_1 - 1 \leq \text{rk}_l \text{Cl}_S(K_1)$$

and

$$\text{rk}_l \text{Cl}_S(K_1) \leq \frac{(l-1)}{r} (\text{rk}_l \text{Cl}_S(K_2) + t_2 + 1) + \frac{l-1}{r} - t_1.$$

Proof. By Lemma 19 we have $H^1(\mu_l) = \mathbb{Z}/l\mathbb{Z}$ and $H^i(\mu_l) = 0$ for $i = 0, 2$. Furthermore $H^0(M) = H^0(N) = 0$. This implies that $H^0(M_k) = 0$ for all k and $H^0(N_k) = 0$ for all k . Additionally we note that by Lemma 20 we have $h^2(M_k) \leq h^1(M_k) - h^0(M_k)$ and since $h^0(M_k) = 0$ we get $h^2(M_k) \leq h^1(M_k)$.

Since cohomology commutes with direct sum we have

$$h^1(N) = \sum_{k=0}^{r-1} h^1(R_k).$$

Now we focus on the filtration 3.4. Taking cohomology of the k th sequence when $k \neq 1$ and $R_k \neq \mathbb{Z}/l\mathbb{Z}$ gives:

$$\begin{array}{ccccccc} 0 & \longrightarrow & 0 & \longrightarrow & 0 & \longrightarrow & 0 \\ & & & & \swarrow & & \\ & & H^1(M_{k-1}) & \longrightarrow & H^1(M_k) & \longrightarrow & H^1(R_k) \\ & & & & \swarrow & & \\ & & H^2(M_{k-1}) & \longrightarrow & H^2(M_k) & \longrightarrow & H^2(R_k) \end{array}$$

and when $k = 1$ gives:

$$\begin{array}{ccccccc} 0 & \longrightarrow & 0 & \longrightarrow & 0 & \longrightarrow & 0 \\ & & & & \swarrow & & \\ & & \mathbb{Z}/l\mathbb{Z} & \longrightarrow & H^1(M_1) & \longrightarrow & H^1(R_1) \\ & & & & \swarrow & & \\ & & 0 & \longrightarrow & H^2(M_1) & \longrightarrow & H^2(R_1) \end{array}$$

and when $R_k = \mathbb{Z}/l\mathbb{Z}$ gives:

$$\begin{array}{ccccccc} 0 & \longrightarrow & 0 & \longrightarrow & 0 & \longrightarrow & 0 \\ & & & & \swarrow & & \\ & & H^1(M_{k-1}) & \longrightarrow & H^1(M_k) & \longrightarrow & \mathbb{Z}/l\mathbb{Z} \\ & & & & \swarrow & & \\ & & H^2(M_{k-1}) & \longrightarrow & H^2(M_k) & \longrightarrow & 0. \end{array}$$

Putting these together gives the desired result. \square

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REFERENCES

- [1] Reinhard Bölling. On ranks of class groups of fields in dihedral extensions over \mathbb{Q} with special reference to cubic fields. *Math. Nachr.*, 135:275–310, 1988.
- [2] Jordan S. Ellenberg and Akshay Venkatesh. Reflection principles and bounds for class group torsion. *Int. Math. Res. Not. IMRN*, (1), 2007.
- [3] Franz Lemmermeyer. Class groups of dihedral extensions. *Math. Nachr.*, 278(6):679–691, 2005.
- [4] James S. Milne. *Etale Cohomology. (PMS-33)*. Princeton University Press, 4 1980.
- [5] James S. Milne. *Arithmetic Duality Theorems*. BookSurge, LLC, second edition, 2006.
- [6] Jacob Tsimerman. Brauer-Siegel for arithmetic tori and lower bounds for Galois orbits of special points. *J. Amer. Math. Soc.*, 25(4):1091–1117, 2012.
- [7] Lawrence C. Washington. *Introduction to Cyclotomic Fields (Graduate Texts in Mathematics)*. Springer, 2nd edition, 12 1996.
- [8] Shou-Wu Zhang. Equidistribution of CM-points on quaternion Shimura varieties. *Int. Math. Res. Not.*, (59):3657–3689, 2005.

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